

An upper bound on the k -modem illumination problem.

Frank Duque * Carlos Hidalgo-Toscano *

October 16, 2014

Abstract

A variation on the classical polygon illumination problem was introduced in [Aichholzer et. al. EuroCG'09]. In this variant light sources are replaced by wireless devices called k -modems, which can penetrate a fixed number k , of “walls”. A point in the interior of a polygon is “illuminated” by a k -modem if the line segment joining them intersects at most k edges of the polygon. It is easy to construct polygons of n vertices where the number of k -modems required to illuminate all interior points is $\Omega(n/k)$. However, no non-trivial upper bound is known. In this paper we prove that the number of k -modems required to illuminate any polygon of n vertices is at most $O(n/k)$. For the cases of illuminating an orthogonal polygon or a set of disjoint orthogonal segments, we give a tighter bound of $6n/k + 1$. Moreover, we present an $O(n \log n)$ time algorithm to achieve this bound.

1 Introduction

The classical art gallery illumination problem consists on finding the minimum number of light sources needed to illuminate a simple polygon. There exist several variations on this problem; one such variation was introduced in [2], it is known as the k -modem illumination problem. For a non-negative number k , a k -modem is a wireless device that can penetrate k “walls”. Let \mathcal{L} be a set of n line segments (or lines) in the plane. A k -modem *illuminates* all points p of the plane such that the interior of the line segment joining p and the k -modem intersects at most k elements of \mathcal{L} . In general, k -modem illumination problems consist on finding the minimum number of k -modems necessary to illuminate a certain subset of the plane, for a given \mathcal{L} . Classical illumination [10, 9] is just the case when $k = 0$.

Several upper bounds have been obtained for various classes of \mathcal{L} . In [2] the authors studied the case when \mathcal{L} is the set of edges of a monotone polygon with

*Departamento de Matemáticas, Cinvestav, D.F. México, México. Partially supported by grant 153984 (CONACyT, Mexico). [frduque, cmhidalgo]@math.cinvestav.mx

n vertices; they showed that the interior of the polygon can be illuminated with at most $\lceil \frac{n}{2k} \rceil$ k -modems ($\lceil \frac{n}{k+4} \rceil$ if $k = 1, 2, 3$), and if the polygon is orthogonal it can be illuminated with $\lceil \frac{n-2}{2k+4} \rceil$ k -modems. In [4] the authors studied the case when \mathcal{L} is a set of n disjoint orthogonal line segments; they showed that $\lceil \frac{n+1}{2(k+1)^{0.264}} \rceil$ k -modems are sufficient to illuminate the plane. In [7] the authors studied the problem of illuminating the plane with few modems of high power; they showed that when \mathcal{L} is an arrangement of lines, one $\lceil \frac{3n}{2} \rceil$ -modem is sufficient, when \mathcal{L} is the set of edges of an orthogonal polygon, one $\lceil \frac{n}{3} \rceil$ -modem is sufficient and when \mathcal{L} is the set of edges of a simple polygon, one $\lceil \frac{2n+1}{3} \rceil$ -modem is sufficient. It is worth noting that there are no published bounds for general polygons.

There are also algorithmic results regarding simple polygons. In [3] the authors presented a hybrid metaheuristic strategy to find few k -modems that illuminate a simple polygon. In this case the k -modems are required to be placed at vertices of the polygon. They applied the hybrid metaheuristic to random sets of simple, monotone, orthogonal and grid monotone orthogonal polygons. Each set consisted of 40 polygons of 30, 50, 70, 100, 110, 130, 150 and 200 vertices. The average numbers of k -modems used by their strategy are shown in Table 1.

	Simple	Monotone	Orthogonal	Grid Monotone	Orthogonal
$k = 2$	$\lceil \frac{n}{26} \rceil$	$\lceil \frac{n}{15} \rceil$	$\lceil \frac{n}{27} \rceil$		$\lceil \frac{n}{18} \rceil$
$k = 4$	$\lceil \frac{n}{52} \rceil$	$\lceil \frac{n}{26} \rceil$	$\lceil \frac{n}{57} \rceil$		$\lceil \frac{n}{35} \rceil$

Table 1: Averages on the number of k -modems obtained by the strategy presented in [3].

It is known that the problem of finding the minimum number of 0-modems to illuminate a simple polygon is NP-hard [1]. It was recently proved that the same problem is also NP-hard for k -modems [reference to the paper of Christiane].

This paper is organized as follows. In Section 2 we present a new bound of $O(n/k)$ for the k -modem illumination problem through a variation of the Cutting Lemma. In Section 3 we present a simple $O(n \log n)$ time algorithm that illuminates a set of n disjoint orthogonal segments with at most $6\frac{n}{k} + 1$ k -modems. This algorithm can be easily modified to obtain the same bound for orthogonal polygons.

2 Upper Bounds Using the Cutting Lemma

A generalized triangle is the intersection of three half planes. Note that a generalized triangle can be a point, a line, a bounded or an unbounded region. Given a set \mathcal{L} of n lines in the plane and $r > 0$, a $\frac{1}{r}$ -cutting is a partition

of the plane into generalized triangles with disjoint interiors, such that each generalized triangle is intersected by at most n/r lines. The Cutting Lemma gives an upper bound on the size of such a cutting.

Theorem 1 (Cutting Lemma). *Let \mathcal{L} be a set of n lines in the plane and $r > 0$. Then there exists a $\frac{1}{r}$ -cutting of size $O(r^2)$.*

The Cutting Lemma was first proved in [5] and independently in [8]. It has become a classical tool in Computational Geometry used mainly in divide-and-conquer algorithms. It can be used to obtain an upper bound on the number of k -modems necessary to illuminate the plane in the presence of n lines.

Theorem 2. *Let \mathcal{L} be a set of n lines in the plane. The number of k -modems required to illuminate the plane in the presence of \mathcal{L} is $O(n^2/k^2)$.*

Proof. By the Cutting Lemma, there exists a $\frac{1}{(n/k)}$ -cutting for \mathcal{L} of size $O(n^2/k^2)$. Note that each triangle can be illuminated with one k -modem. Thus the number of k -modems required to illuminate the plane is $O(n^2/k^2)$. \square

Given a polygon \mathcal{P} we can obtain the same bound on the number of k -modems needed to illuminate it. We first extend its edges to straight lines and then apply Theorem 2. We can achieve a better upper bound of $O(n/k)$ by making use of a line segment version of the Cutting Lemma given in [6]. In that paper the authors consider cuttings in a more general setting: they define a cutting as a subdivision of the plane in *boxes*. A *box* is a closed subset of the plane which has constant description (that is, it can be represented in a computer with $O(1)$ space, and it can be checked in constant time whether a point lies in a box or whether an object intersects (the interior of) a box).

Theorem 3. [6] *Let \mathcal{L} be a set of n line segments in the plane with a total of A intersections and $r > 0$. Then there exists a $\frac{1}{r}$ -cutting for \mathcal{L} of size $O\left(r + A\left(\frac{r}{n}\right)^2\right)$.*

If we consider a polygon as a set of n line segments that intersect only at their endpoints, Theorem 3 gives us a $\frac{1}{(n/k)}$ -cutting of size $O\left(\frac{n}{k} + n\left(\frac{n^2}{k^2} \frac{1}{n^2}\right)\right) = O\left(\frac{n}{k}\right)$. Taking into account that the boxes used in the proof of Theorem 3 are generalized trapezoids (that is, intersections of four half planes), and that it is possible to illuminate each trapezoid with a k -modem, we obtain Theorem 4. Note that the same reasoning applies to sets of segments in the plane, which gives a bound of $O(n/k + A/k^2)$ for that case.

Theorem 4. *Let \mathcal{P} be a polygon with n vertices. The number of k -modems needed to illuminate \mathcal{P} is at most $O(n/k)$.*

3 An Algorithm for Orthogonal Line Segments Illumination

In this section we present an $O(n \log n)$ time algorithm to illuminate the plane with k -modems in the presence of a set \mathcal{L} of n disjoint orthogonal segments. The number of k -modems used by our algorithm is at most $6\frac{n}{k} + 1$.

We assume that \mathcal{L} is contained in a rectangle R . Our objective is to partition R into a certain kind of polygons called staircases, in a similar fashion to the cuttings introduced in Section 2. We do this in such a way so that each staircase is illuminable with one k -modem. A *staircase* is an orthogonal polygon P such that: P is bounded from below by a single horizontal segment $Floor(P)$; P is bounded from the right by a single vertical segment $Rise(P)$; the left endpoint of $Floor(P)$ and the upper endpoint of $Rise(P)$ are joined by a monotone polygonal chain $Steps(P)$. See Figure 3.1. In what follows let P be a staircase.

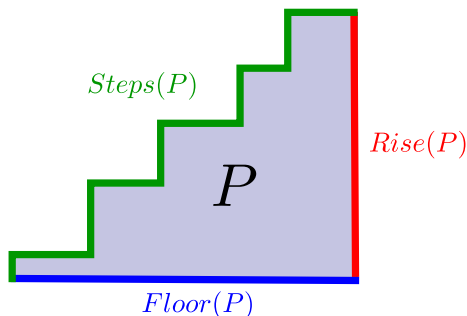


Figure 3.1: An example of a staircase.

Note that an axis parallel line intersecting the interior of P splits it into two parts which are also staircases. Let l be a horizontal line that intersects the interior of P in a segment s . We denote by $Above(P, l)$ the staircase formed by s and the part of P above s , and denote by $Below(P, l)$ the staircase formed by s and the part of P below s . Likewise, let l' be a vertical line that intersects the interior of P in a segment s . We denote by $Left(P, l')$ the staircase formed by s and the part of P to the left of s , and denote by $Right(P, l')$ the staircase formed by s and the part of P to the right of s . See Figure 3.2.

3.1 Illumination Algorithm

Let \mathcal{P} be the set of endpoints of the segments in \mathcal{L} , excepting the rightmost points of the horizontal segments. The algorithm *StaircasePartition*(\mathcal{L}) finds a rectangle R that contains \mathcal{L} and produces as output a partition of R into staircases. Initially, R is the only staircase in the partition, we use a horizontal sweep line l starting at the top of R that stops at each point of \mathcal{P} and checks whether it is necessary to refine the partition.

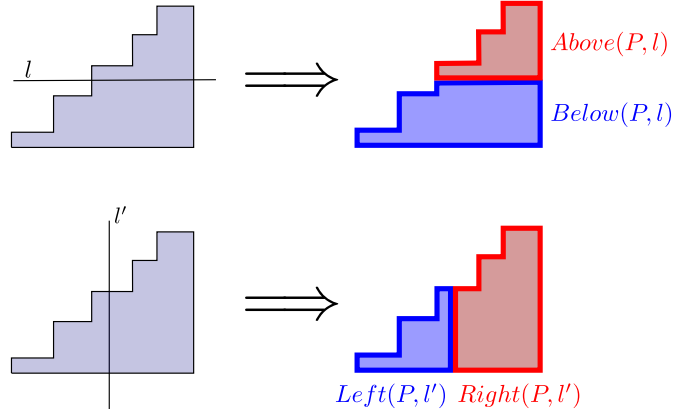


Figure 3.2: An example of $Above(P, l)$, $Below(P, l)$, $Left(P, l')$ and $Right(P, l')$.

Throughout the algorithm, the staircases that are not intersected by the sweep line need no further processing. The staircases that are intersected by the sweep line might be splitted later. There are two reasons to split a staircase: the number of segments of \mathcal{L} that intersect it might be too high, or a horizontal segment of \mathcal{L} might cross the staircase completely. These two cases are handled by procedures called *OverflowCut* and *CrossingCut* respectively. *OverflowCut* ensures that no staircase is intersected by more than k segments; *CrossingCut* ensures that no horizontal segment in \mathcal{L} crosses completely a staircase.

OverflowCut is called whenever k segments intersect the interior of a staircase P . *OverflowCut* starts by replacing P by $Above(P, l)$. If at most $\lfloor \frac{k}{2} \rfloor$ segments intersect both P and l , $Below(P, l)$ is added to the partition. If more than $\lfloor \frac{k}{2} \rfloor$ segments intersect both P and l , we search for a vertical line l' that leaves at most $\lfloor \frac{k}{2} \rfloor$ of those segments to each side. The staircases $Left(Below(P, l), l')$ and $Right(Below(P, l), l')$ are added to the partition. See Figure 3.3.

CrossingCut is called whenever a horizontal segment s of \mathcal{L} intersects at least three staircases of the partition. Let P_1, \dots, P_m be these staircases ordered from left to right. *CrossingCut* replaces each P_i by $Above(P_i, l)$ for $2 \leq i \leq m-1$, and merges $Below(P_i, l)$ with P_m . Note that the number of staircases in the partition does not grow. See Figure 3.4.

The algorithm begins by sorting the points in \mathcal{P} by y coordinate, which will be the stop points for the sweep line. After initializing the partition with R , the stop points are processed according to three cases. (As part of the steps taken in each case, we maintain the set of segments that intersect each staircase in the partition sorted from left to right, making a distinction of the ones that also intersect l .)

- The sweep line stops at an upper endpoint. This means we have found a new vertical segment that intersects a staircase P . That staircase may now be intersected by k segments, if this happens we make an *OverflowCut*.

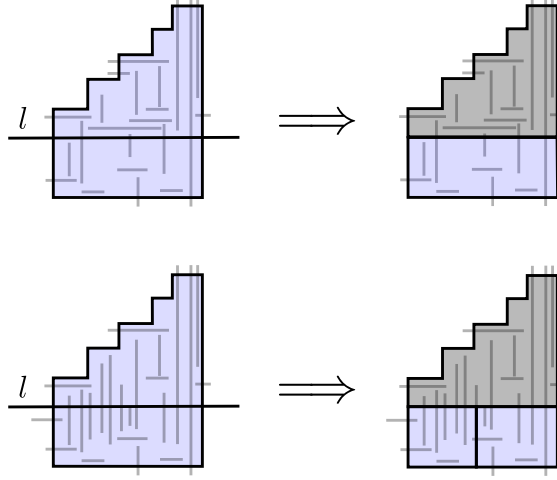


Figure 3.3: The two cases for an OverflowCut on a staircase with $k = 18$.

- The sweep line stops at a lower endpoint. This means we have found the end of a segment that intersects P . In this case we only update the set of P .
- The sweep line stops at a left endpoint p of a segment s . If the number of staircases that s intersects is at least three, we perform a *CrossingCut* on them. If necessary, an *OverflowCut* is made on the staircases that contain the endpoints of s .

At the end of the algorithm, the interior of each staircase P in the partition is intersected by at most k segments. Thus, a k -modem placed in the intersection between $Floor(P)$ and $Rise(P)$ is enough to illuminate it.

Lemma 5. *Let \mathcal{L} be a set of n disjoint orthogonal segments. Let \mathcal{P}' be the set of endpoints of the segments in \mathcal{L} , excepting the lower endpoints of the vertical segments. Then, the total number of calls to **OverflowCut** made by **StaircasePartition**(\mathcal{L}) is at most $|\mathcal{P}'| / \lceil k/2 \rceil$.*

Proof. Let P be a staircase in the partition that **StaircasePartition**(\mathcal{L}) returns. P was created after executing an *OverflowCut*, so $Steps(P)$ was initially intersected by at most $\lfloor k/2 \rfloor$ vertical segments. Further modifications to P are done by *CrossingCuts*, which add edges to $Steps(P)$. The horizontal edges added to $Steps(P)$ are part of segments in \mathcal{L} , so they can't be intersected by any other segment. Thus, at most $\lfloor k/2 \rfloor$ vertical segments of \mathcal{L} cross completely P .

If *OverflowCut* is called on P , $Above(P, l)$ becomes a staircase of the partition and won't be modified anymore. *OverflowCut* was called because k segments from \mathcal{L} intersected P . These same segments intersect also $Above(P, l)$; of these at most $\lfloor k/2 \rfloor$ cross it completely. Therefore, at least $\lceil k/2 \rceil$ points of \mathcal{P}' are

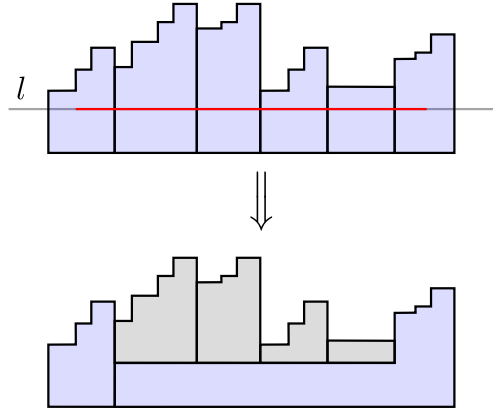


Figure 3.4: A CrossingCut.

contained in $Above(P, l)$. Thus, the total number of *OverflowCut* calls is at most $|\mathcal{P}'| / \lceil k/2 \rceil$.

□

Theorem 6. *Let \mathcal{L} be a set of n orthogonal disjoint segments. Then the number of k -modems needed to illuminate the plane in the presence of \mathcal{L} is at most $6\frac{n}{k} + 1$. The locations for those modems can be found in $O(n \log n)$ time.*

Proof. We can assume that the number of vertical segments in \mathcal{L} is at least $n/2$, otherwise we can rotate the plane. Let \mathcal{P}' be the set of endpoints of the segments in \mathcal{L} , excepting the lower endpoints of the vertical segments. Therefore, $|\mathcal{P}'| \leq 3n/2$. *StaircasePartition*(\mathcal{L}) adds staircases to the partition only when it makes a call to *OverflowCut*. The staircases added are at most two. Using Lemma 5, we obtain that there are at most $2|\mathcal{P}'| / \lceil k/2 \rceil + 1 \leq 6\frac{n}{k} + 1$ staircases in the partition. Since each staircase is illuminable with one k -modem, the bound follows.

It remains to prove the $O(n \log n)$ bound on the time to find the locations of the k -modems.

For each staircase P in the partition, we maintain the horizontal segments of $Steps(P)$ sorted from left to right and the vertical segments of $Steps(P)$ sorted from bottom to top. Thus it is possible to find $Above(P, l)$ and $Below(P, l)$ in $O(\log n)$ time; the same happens with $Left(P, l')$ and $Right(P, l')$. An *OverflowCut* call makes at most two splittings, so it takes at most $O(\log n)$ time. Since the number of *OverflowCut* calls is $O(n/k)$, the total time required by them is $O(\frac{n}{k} \log n)$.

Given m staircases P_1, \dots, P_m intersected by a segment s , the splits and merges done by a *CrossingCut* can be achieved in $(m - 2)O(\log n)$ time. The $m - 2$ upper parts of the staircases splitted horizontally become staircases that won't be modified again. Since there are at most $O(n/k)$ staircases, at most $O(n/k)$ splits and merges from *CrossingCut* are done throughout the algorithm.

Since each pair of split and merge operations takes $O(\log n)$ time, the total time required by the *CrossingCut* calls is $O(\frac{n}{k} \log n)$.

Besides the cost of the calls to *OverflowCut* and *CrossingCut*, at every stop point, we must determine the staircase where it is located. This can be done in $O(\log n)$ time by maintaining the order from left to right in which the sweep line intersects the staircases. Thus, the running time for the algorithm is $O(n \log n + \frac{n}{k} \log n)$. \square

4 Acknowledgements

We would like to thank Ruy Fabila-Monroy for introducing us to the k -modem problem and for his invaluable help in the development of this paper.

References

- [1] A. Aggarwal. *The Art Gallery Theorem: Its Variations, Applications and Algorithmic Aspects*. PhD thesis, 1984. AAI8501615.
- [2] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, J. Urrutia, and B. Vogtenhuber. Modem Illumination of Monotone Polygons. In *Proc. 25th European Workshop on Computational Geometry EuroCG '09*, pages 167–170, Brussels, Belgium, 2009.
- [3] A.L. Bajuelos, S. Canales, G. Hernández, and M. Martins. A hybrid meta-heuristic strategy for covering with wireless devices. *J.UCS*, 18(14):1906–1932, 2012.
- [4] B. Ballinger, N. Benbernou, P. Bose, M. Damian, E. D. Demaine, V. Dujmović, R. Flatland, F. Hurtado, J. Iacono, A. Lubiw, P. Morin, V. Sacristán, D. Souvaine, and R. Uehara. Coverage with k -transmitters in the presence of obstacles. In Weili Wu and Ovidiu Daescu, editors, *Combinatorial Optimization and Applications*, volume 6509 of *Lecture Notes in Computer Science*, pages 1–15. Springer Berlin Heidelberg, 2010.
- [5] B. Chazelle and J. Friedman. A deterministic view of random sampling and its use in geometry. *Combinatorica*, 10(3):229–249, 1990.
- [6] M. de Berg and O. Schwarzkopf. Cuttings and applications. *International Journal of Computational Geometry & Applications*, 5(04):343–355, 1995.
- [7] R. Fabila-Monroy, A.R. Vargas, and J. Urrutia. On modem illumination problems. In *XIII Encuentros de Geometría Computacional*, Zaragoza, Spain, 2009.
- [8] J. Matoušek. Construction of epsilon nets. In *Proceedings of the Fifth Annual Symposium on Computational Geometry, SCG '89*, pages 1–10, New York, NY, USA, 1989. ACM.

- [9] J. O’rourke. *Art gallery theorems and algorithms*, volume 57. Oxford University Press Oxford, 1987.
- [10] J.R. Sack and J. Urrutia. *Handbook of computational geometry*. Elsevier, 1999.